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Stability of quantum oscillators

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Abstract. The quasi-energies of certain periodically forced quantum anharmonic oscillators have no absolutely continuous spectrum.

1. Introduction

Let H be a discrete quantum Hamiltonian on $\mathcal{H}$, with simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Let $V(t)$ be a perturbing energy, 2π-periodic in t

$$V(t+2\pi) = V(t)$$

and define the *quasi-energy* to be

$$K = -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t} + H + V(t)$$

on $L_2(0, 2\pi) \otimes \mathcal{H}$ with periodic boundary conditions in t.

Assume that $V(t)$ is *bounded* and strongly C^r in t, and that the spectrum of H has *increasing gaps*

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n \geqslant cn^\gamma \tag{1.1}$$

for some $c > 0$ and $\gamma > 0$. Under these conditions, one has

Theorem 1 [2]. The quasi-energy K has no absolutely continuous spectrum, provided $r \geqslant [\gamma^{-1}] + 1$.

The question naturally arises as to the necessity of the boundedness of $V(t)$, and the gap condition (1.2). The *forced harmonic oscillator*

$$\frac{1}{2}\,p^2 + \frac{\omega^2}{2}\,x^2 + \beta x \sin t$$

is an explicitly integrable system [1], which is known to be dense pure point, except for $\omega = 1$, in which case it is *absolutely continuous*. The hypothesis of theorem 1 fails on both counts, since x is unbounded, and the gap exponent $\gamma = 0$.

On the other hand, the *forced anharmonic oscillator*

$$\frac{1}{2}\,p^2 + \frac{\omega^2}{2}\,x^2 + gx^4 + \beta x \sin t \tag{1.2}$$

satisfies a gap condition. For by Bohr–Sommerfeld (for example), the eigenvalues of the anharmonic oscillator satisfy

$$\lambda_n \sim n^4/3$$

so that $\gamma = \frac{1}{3}$. The perturbation is, of course, again unbounded.

We shall show that the quasi-energy K for (1.2) is purely singular if β is sufficiently small. In addition, if the anharmonic term x^4 is replaced by x^{2p} with $p > 2$, then K is purely singular for *all* β.

The method is an adaptation of the adiabatic analysis of [2, part I] which shows, roughly, that a relatively H-bounded perturbation $V(t)$ may be replaced by a bounded perturbation at the cost of one time derivative.

2. Adiabatic analysis

Let H be discrete self-adjoint, with $\lambda_0 < \lambda_1 < \cdots$, and $V(t)$ a 2π-periodic family of relatively H-bounded operators. Denote by K the quasi-energy

$$K = -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t} + H + V(t)\,.$$

Define the gap

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n$$

and let Γ_n be the circle with centre λ_n and radius

$$r_n = \tfrac{1}{4}\min\{\Delta\lambda_n, \Delta\lambda_{n-1}\}\,.$$

Define M_n by

$$M_n = \sup\{\|V(t)(H-z)^{-1}\| : z \in \Gamma_n,\ 0 \leqslant t \leqslant 2\pi\}\,.$$

Theorem 2. Assume that all eigenvalues of H are simple (i.e. non-degenerate), and that

$$\lim M_n = 0\,. \tag{2.1}$$

If $V(t)(H+1)^{-1}$ is strongly C^{r+1} in t, for some $r \geqslant 0$, then K is unitarily equivalent to an operator

$$K_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + H_1 + V_1(t)$$

where $V_1(t)$ is bounded in C^r in t, and H_1 is diagonal in the same basis as H, with eigenvalues $\lambda_n^{(1)}$ satisfying

$$|\lambda_n^{(1)} - \lambda_n| \leqslant r_n\,.$$

Proof. The proof follows the adiabatic method of [2, part I]. Let $H(t) = H + V(t)$, $R(z,t) = (H(t) - z)^{-1}$, and $R_0(z) = (H-z)^{-1}$. It is easily shown that $R(z,t)$ is strongly C^{r+1} of z is in the resolvent set of $H(t)$.

By (1.1), there exists N such that for $n \geqslant N$,

$$M_n \leqslant \tfrac{1}{3}\,. \tag{2.2}$$

It therefore follows from

$$R(z,t) - R_0(z) = R_0(z) \sum_{n=1}^{\infty} [V(t) R_0(z)]^n$$

that on Γ_n

$$\|R(z,t) - R_0(z)\| \leqslant \tfrac{3}{2} r_n^{-1} M_n\,. \tag{2.3}$$

Defining

$$P_n(t) = (2\pi \mathrm{i})^{-1} \int_{\Gamma_n} R(z,t)\mathrm{d}z$$

we obtain

$$\|P_n(t) - P_n(0)\| \leqslant \tfrac{3}{2} M_n \leqslant \tfrac{1}{2}\,.$$

It follows that $\dim P_n(t) = \dim P_n(0) = 1$, so that there is a unique eigenvalue $\lambda_n(t)$ of $H_\beta(t)$ inside Γ_n. Again, $P_n(t)$ and $\lambda_n(t)$ are C^{r+1} in t.

Therefore, if one defines the unitary transformation $U(t)$ as in [2], that is, by the first equation following (5.17) on p 319 of [2], then $U(t)$ is strongly C^{r+1} in t, and

$$U(t) K U^*(t) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + H_1(t) - \mathrm{i}U(t)\dot{U}^*(t)$$

where $H_1(t) = \mathrm{diag}\{\lambda_n^{(1)}(t)\}$ is diagonal in the same basis as H.

An elementary gauge transformation [2, p 322], then replaces $\lambda_n^{(1)}(t)$ by its mean

$$\lambda_n^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} \lambda_n^{(1)} \mathrm{d}t\,.$$

The new $V_1(t)$ is just $-\mathrm{i}\,U(t)\dot{U}^*(t)$ (gauge-transformed), which is clearly C^r. □

Corollary 1. If it is assumed only that

$$M = \sup M_n < \infty \tag{2.4}$$

then the same result holds for

$$H_\beta(t) = H + \beta V(t)$$

if β is sufficiently small.

Proof. The proof of theorem 2 requires only that

$$M_n \leqslant \tfrac{1}{3}\,.$$

If $V(t)$ is replaced by $\beta V(t)$, then M_n is replaced by

$$|\beta| M_n \leqslant |\beta| M\,.$$

So it is enough to require that

$$|\beta| \leqslant \frac{1}{3M}\,.$$ □

Remark. These results may be generalized to non-simple (degenerate) eigenvalues by working with block matrices as in the appendix of [3].

3. Anharmonic oscillators

We shall consider the Hamiltonian

$$H(t)=\frac{1}{2}p^2+\frac{\omega^2}{2}x^2+gx^{2p}+\beta x\sin t$$

where $g>0$ and $p\geqslant 1$.

Theorem 3. The quasi-energy of $H(t)$ is purely singular if either (*a*) $p>1$, or (*b*) $p=1$ and β is sufficiently small.

Lemma 1. Let A and B be positive self-adjoint, and $H=A+B+1$ the form sum. Then for $0\leqslant\alpha\leqslant\frac{1}{2}$, the operator $B^\alpha H^{-\alpha}$ is bounded.

Proof. We need to show that

$$\operatorname{ran}H^{-\alpha}=D(H^\alpha)\subset D(B^\alpha).$$

This is clear if $\alpha=0$ and $\frac{1}{2}$, and therefore holds by interpolation. □

Corollary 2. The operator

$$x(p^2+x^2+gx^{2p}+1)^{-s}$$

is bounded if $p\geqslant 1$ and $s=1/2p$, where $g>0$.

Proof. Let $B=|x|$, and $A=p^2+x^2$. □

Proof of Theorem 3. The eigenvalues of

$$H=\frac{1}{2}p^2+\frac{\omega^2}{2}x^2+gx^{2p}$$

satisfy

$$\lambda_n\sim c\,n^{2p/p+1} \tag{3.1}$$

by Bohr–Sommerfeld. To estimate M_n, with $V(t)=Fx\sin t$, we need to estimate

$$x(H-z)^{-1}=[x(H+1)^{-1/2p}][(H+1)^{1/2p}(H-z)^{-1}].$$

The first factor is bounded by corollary 2. For $z\in\Gamma_n$, $(H-z)^{-1}$ has norm $4r_n^{-1}$, where

$$r_n\sim\frac{\lambda_n}{n}\sim c\,n^{(p-1)/(p+1)} \tag{3.2}$$

while $(H+1)^{1/2p}$ is of order

$$(\lambda_n)^{1/2p}\sim n^{1/(p+1)}.$$

Therefore

$$M_n\sim n^{1/(p+1)}n^{-(p-2)/(p+1)}$$

so that

$$\lim M_n=0$$

if $p>2$, while M_n is bounded for $p=2$.

The result now follows by applying first theorem 2 to reduce to a bounded perturbation, and then theorem 1, with gap exponent

$$\gamma=\frac{p-1}{p+1}>0.$$

Part (*b*) uses corollary 1. □

Remark. A generalization is perhaps worth remarking. If $x \sin t$ is replaced by $v(x) \sin t$, where $v(x) \sim |x|^q$ at infinity, then the same argument shows that

$$M_n \sim (\lambda_n)^{q/2p} r_n^{-1} \sim n(1 + q - p)/(p + 1) . \tag{3.3}$$

Thus M_n *tends to zero if*

$$p > 1 + q . \tag{3.4}$$

Therefore, if absence of an absolutely continuous spectrum is considered to be 'stability', then the x^4 anharmonic oscillator is stable under

$$\beta |x|^\alpha \sin t$$

for $0 < \alpha < 1$, and all β.

Similarly, if $\omega(t)$ is a smooth periodic function, the quasi-energy of

$$\frac{1}{2} p^2 + \frac{\omega^2(t)}{2} x^2 + g x^{2p}$$

is purely singular provided that $p > 3$.

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References

[1] Hagedorn G, Loss M and Slawny J 1986 Non-stochasticity of time-dependent quadratic Hamiltonians and the spectra of transformations *J. Phys. A: Math. Gen.* **19** 521–31

[2] Howland J S 1989 Floquet operators with singular spectrum, I *Ann. Inst. H Poincare* **50** 309–23; II *Ann. Inst. H Poincare* 325–34

[3] Howland J S 1991 Quantum stability *Proc. Scattering Theory Workshop (Aarhus, Denmark)* (Berlin: Springer) in press